ON THE METHODS OF DUAL INTEGRAL EQUATIONS AND DUAL SERIES AND THEIR APPLICATION TO PROBLEMS OF MECHANICS

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In the paper dual integral equations and dual series of a general type are considered, widely used for the solution to boundary value problems in the theory of elasticity, hydrodynamics, electrostatics etc., under mixed boundary conditions. A method of solution to dual equations and series is proposed which is based on their reduction to Fredholm integral equations of the second kind with respect to an orthogonalizing kernel. For this purpose linear integral equations of the Volterra type, being a continual analog of the orthogonalization process, are used. The application of this method is illustrated by the plane contact problem for a wedge.

1. The method of dual integral equations and dual series is one of the most effective means to solve boundary value problems with mixed boundary conditions. Dual equations and series are usually applied in those cases, when the solution to the boundary value problem is sought in the form of an expansion of a certain system of functions, and when mixed boundary conditions are used for the determination of the coefficients of the expansion. As a result of application to the solution of different operators on diferent parts of the boundary, dual equations or dual series are obtained. In the case of real expansions they may be represented in the following general form:

$$\int_{-\infty}^{\infty} [1 + h(\xi)]\,\psi(\xi)\,u(\xi, \eta)\,d\tau(\xi) = g_1(\eta) \qquad (-\infty < a < \eta < c)$$

$$\int_{-\infty}^{\infty} \psi(\xi)\,u(\xi, \eta)\,d\tau(\xi) = g_2(\eta) \qquad (c < \eta < b < \infty) \qquad (1.1)$$

Here all functions except $\psi(\xi)$ are known; $\tau(\xi)$ is a spectral distribution function of the corresponding expansion. For expansion in integrals $\tau(\xi)$ is continuous, for expansion in series it is represented as a step function with a countable number of jumps.

First, apparently, the problem of dual integral equations was formulated by Weber [1] in 1873 and solved (for a very special case) by Beltrami [2] in 1881. The second birth of dual equations occurred after Titchmarsh [3] and Busbridge [4] on the basis of the theory of Mellin transforms, gave the solution by quadratures to equations with Bessel kernels

$$\begin{aligned} &\int_0^\infty r(\xi) J_\nu(\xi\eta)\psi(\xi)\,d\xi = g_1(\eta) \qquad (0<\eta<1) \\ &\int_0^\infty J_\nu(\xi\eta)\psi(\xi)\,d\xi = g_2(\eta) \qquad (1<\eta<\infty) \end{aligned} \tag{1.2}$$

for the case $r(\xi) = \xi^\alpha$, $g_2(\eta) \equiv 0$. Later, several papers appeared, in which by means of various methods, dual equations of such kind were studied in considerable detail (among them for $g_2 \not\equiv 0$). Results of a more general character were obtained in investigation, dealing with dual equations with Bessel kernels for arbitrary $r(\xi)$, when, in general, one is not successful to find a solution by quadratures. Thus, in paper [5] the solution of dual equations reduces to the solution of an infinite system of linear algebraic equations, and in [6 to 9] they are reduced by various methods to Fredholm integral equations of the second kind. The method of Weiner-Hopf-Fok and variational method of solving dual equations are applied by Noble [8 to 10]. Some special cases of $r(\xi)$ are considered in [11 and 12].

In separate papers dual equations with other kernels are examined (*) in [13] – with a kernel in the form of a Legendre function with a complex power (the kernel of the Mehler-Fok integral transform), in [14] – with a kernel in the form of a complex cylindrical function of the first and second kind (kernel of Weber transform). The investigation of dual equations with kernels of Fourier transforms may be found in the monograph [15]. Dual series for different functions (trigonometric, cylindrical, Legendre functions, Jacobi functions) were investigated in papers [16 to 21] and others.

In the present paper basic consideration is given to dual integral equations and dual series of the type (1.1). It will be shown that there exist very general methods of reduction of dual equations (1.1) to a single integral equation, defined on the whole interval (a, b) and allowing inversion. This method is connected with the continual orthogonalization of the integrand functions in dual equations and its idea is quite close to the known method of solution to the inverse Sturm-Liouville problem developed in the fundamental investigations [22 and 23]. Since the paper pursues primarily applied goals, the formal side of the general method will be mainly exposed here, while the question of all necessary conditions and possible restrictions may be determined by investigation of specific equations with these or other kernels.

2. Setting $1 + h(\xi) = \rho^2(\xi)$, $\rho(\xi)\psi(\xi) = f(\xi)$, we symmetrize dual equations (1.1), reducing them to the form convenient for further operations

$$\begin{aligned} &\int_{-\infty}^\infty \rho(\xi) f(\xi) u(\xi, \eta)\, d\tau(\xi) = g_1(\eta) \qquad (a<\eta<c) \\ &\int_{-\infty}^\infty \rho^{-1}(\xi) f(\xi) u(\xi, \eta)\, d\tau(\xi) = g_2(\eta) \qquad (c<\eta<b) \end{aligned} \tag{2.1}$$

For the solution of Equations (2.1) we attempt to orthogonalize the functions $\rho(\xi)u(\xi, \eta)$ and $\rho^{-1}(\xi)u(\xi, \eta)$, such that the result of orthogonalization will lead to one and the same kernel, which corresponds to the spectral

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*) Particular dual integral equations with kernels, depending upon difference in arguments are studied in the papers of F.D. Gakhov, I.Ts. Gokhberg, M.G. Krein, I.M. Rapoport, Iu.I. Cherkesov and others.

distribution function $\tau(\xi)$

It is known that with the aid of linear triangular transform one may orthogonalize any system of linearly independent functions. A continual analog of this process is represented by the linear integral transformation of the Volterra type. In the general case the problem of construction of integral transformation of a given function of two variables into an orthogonal kernel, apparently, as yet, has no solution. The particular case of orthogonalization of the function $\cos\sqrt{\lambda x}$ with respect to measures x and $\rho(\lambda)$ was investigated by Gel'fan and Levitan in paper [22], dealing with the restoration of a second order differential operator from its spaectral characteristics.

We consider at first the case when the kernel of Equation (2.1) is orthogonal, i.e. a continuous function $u(\xi, \eta)$ and nondecreasing functions $\tau(\xi)$, $\sigma(x)$ are such that Formulas

$$F(\xi) = \int_a^b f(x)\, u(\xi, x)\, d\sigma(x), \quad f(x) = \int_{-\infty}^{\infty} F(\xi)\, u(\xi, x)\, d\tau(\xi) \tag{2.2}$$

establish a mutually inverse isometric mapping of spaces $L_{2,\sigma}$ of all σ-measurable functions $f(x)$ $(a \le x < b)$ having σ-integrable square

$$\int_a^b |f(x)|^2\, d\sigma(x) < \infty$$

on the space $L_{2,\tau}$ of τ-measureable functions $F(\xi)$ $(-\infty < \xi < \infty)$, havihg τ-square integrable on the whole axis. We suppose also that $\rho(\xi)$ is continuous and $\rho(\xi)$, $\sigma_1(\eta)$ and $\sigma_2(\eta)$ are sufficiently regular in order that the integrals considered below exist at least in the sense of generalized functions.

Relation (2.2) determines integral transforms with finite or infinite limits (i.e. expansions by certain systems of functions), the applications of which for the solution of boundary value problems leads to the dual equations (2.1).

Using formulas obtained from (2.2) by changing $u(\xi, x)$ for $\varphi(\xi, x)$ we construct a kernel which reflects the spaces $L_{2,\sigma}$ and $L_{2,\tau}$ one on another, and is associated with $u(\xi, x)$ by relation

$$\varphi(\xi, x) = \rho(\xi)\left[u(\xi, x) + \int_a^x K(x, \eta)\, u(\xi, \eta)\, d\sigma(\eta)\right] \tag{2.3}$$

where $K(x, \eta)$ $(\eta < x)$ is a certain unknown continuous function. We can consider Expression (2.3) as a Volterra equation relative to $u(\xi, x)$ and its solution, i.e. function $u(\xi, x)$ may be represented in the form

$$u(\xi, x) = \rho^{-1}(\xi)\left[\varphi(\xi, x) + \int_a^x H(x, \eta)\, \varphi(\xi, \eta)\, d\sigma(\eta)\right] \tag{2.4}$$

Continuous functions $K(x, \eta)$ and $H(x, \eta)$ $(\eta < x)$ are called orthogonalized kernels. We consider now the integral

$$H_0(x, \eta) = \int_{-\infty}^{\infty} \rho(\xi)\, u(\xi, \eta)\, \varphi(\xi, x)\, d\tau(\xi) \tag{2.5}$$

If $\rho(\xi)u(\xi,\eta) \in L_{2,\tau}$, when $H_0(x,\eta) \in L_{2,\sigma}$. Introducing into consideration generalized functions, we can extend u- and φ-transforms, realized by Equations (2.2) also to functions not belonging to $L_{2,\sigma}$ and $L_{2,\tau}$; for example, functions integrable (by corresonding measure) and of bounded variation on any finite interval. Thus, in this case when $\rho(\xi)u(\xi,\eta) \overline{\in} L_{2,\tau}$, $H_0(x,\eta)$ may be considered as a generalized function, the φ-transform of which has the form

$$\int_a^b H_0(x,\eta)\varphi(\xi,x)\,d\sigma(x) = \rho(\xi)u(\xi,\eta) \tag{2.6}$$

Comparing (2.4) with (2.6), we obtain

$$H_0(x,\eta) = [\sigma'(\eta)]^{-1}\delta(x-\eta) + \begin{cases} H(\eta,x) & (x<\eta) \\ 0 & (x>\eta) \end{cases} \tag{2.7}$$

where $\delta(x-\eta)$ is a delta function; $\sigma'(\eta)$ in the general case is understood as a generalized function. On the other hand, from (2.5) and (2.2) we have

$$\rho(\xi)\varphi(\xi,x) = \int_a^b H_0(x,\eta)u(\xi,\eta)\,d\sigma(\eta) \tag{2.8}$$

Hence

$$\varphi(\xi,x) = \rho^{-1}(\xi)\left[u(\xi,x) + \int_x^b H(\eta,x)u(\xi,\eta)\,d\sigma(\eta)\right] \tag{2.9}$$

Substituting now into (2.5) the function $\varphi(\xi,x)$, from (2.3) we get

$$\Psi_0(\eta,x) + \int_a^x K(x,\eta_1)\Psi_0(\eta,\eta_1)\,d\sigma(\eta_1) = [\sigma'(\eta)]^{-1}\delta(x-\eta) + \begin{cases} H(\eta,x) & (x<\eta) \\ 0 & (x>\eta) \end{cases} \tag{2.10}$$

where

$$\Psi_0(\eta,x) = \int_{-\infty}^{\infty} \rho^2(\xi)u(\xi,\eta)u(\xi,x)\,d\tau(\xi) \tag{2.11}$$

As a consequence of (2.2)

$$\int_{-\infty}^{\infty} u(\xi,x)u(\xi,\eta)\,d\tau(\xi) = [\sigma'(\eta)]^{-1}\delta(x-\eta) \tag{2.12}$$

therefore, introducing the new function

$$\Psi(\eta,x) = \int_{-\infty}^{\infty} h(\xi)u(\xi,\eta)u(\xi,x)\,d\tau(\xi) \tag{2.13}$$

from (2.10) for values of $x > \eta$ of interest to us, we obtain

$$\Psi(\eta,x) + \int_a^x K(x,\eta_1)\Psi(\eta,\eta_1)\,d\sigma(\eta_1) + K(x,\eta) = 0 \tag{2.14}$$

The kernel $\Psi(\eta,\eta_1)$ of the integral equation (2.14) may be expressed by means of the orthogonalization kernel $H(x,\eta)$. Using expressions (2.4) and (2.5) in (2.13) (when $x > \eta$), we find

$$\int_{-\infty}^{\infty} [\rho^2(\xi)-1]u(\xi,x)u(\xi,\eta)\,d\tau(\xi) = \int_{-\infty}^{\infty} \rho(\xi)u(\xi,\eta)\Big[\varphi(\xi,x) + \int_a^x H(x,\eta_1)\varphi(\xi,\eta_1)\,d\sigma(\eta_1)\Big]d\tau(\xi) - [\sigma'(\eta)]^{-1}\delta(x-\eta) = H_0(x,\eta) + \int_a^x H(x,\eta_1)H_0(\eta_1,\eta)\,d\sigma(\eta_1) - [\sigma'(\eta)]^{-1}\delta(x-\eta) \tag{2.15}$$

Taking into consideration (2.7), we may now write

$$\Psi(\eta, x) = \int_a^\eta H(x, \eta_1)\{[\sigma'(\eta)]^{-1}\delta(\eta-\eta_1) + H(\eta, \eta_1)\}\, d\sigma(\eta_1) \quad (2.16)$$

Hence

$$\Psi(\eta, x) = H(x, \eta) + \int_a^\eta H(x, \eta_1) H(\eta, \eta_1)\, d\sigma(\eta_1) \quad (2.17)$$

It follows from (2.17) that $\Psi(\eta, x)$ is continuous, so long as the kernel $H(x, \eta)$ is continuous. Thus, for every fixed x, Equation (2.14) is a linear Fredholm integral equation of the second kind with continuous, symmetrical kernel $\Psi(\eta, x)$. This equation, just as the nonlinear integral equation (2.17) with respect to the orthogonalization kernel $H(x, \eta)$ is analogous to the integral equation studied in detail in [22 and 23]. In our case these equations, although being of a rather more general character, are essentially little different from considerations in [22 and 23], therefore, using the close analogy between them, it is not difficult to prove solvability of Equation (2.14).

We shall prove first that if $f(\eta) \in L_{2,\sigma}$ is some finite function and

$$\Phi(\xi) = \int_a^b f(\eta)\, u(\xi, \eta)\, d\sigma(\eta)$$

is its u-transform, then from the equality

$$\int_{-\infty}^{\infty} \Phi^2(\xi)\, \rho^2(\xi)\, d\tau(\xi) = 0 \quad (2.18)$$

it necessarily follows that almost everywhere $\Phi(\xi) \equiv 0$, i.e. $f(\eta) \equiv 0$. We have

$$\rho(\xi)\Phi(\xi) = \int_a^b f(\eta)\, d\sigma(\eta) \int_a^b H_0(\eta_1, \eta)\, \varphi(\xi, \eta_1)\, d\sigma(\eta_1) =$$

$$= \int_a^b \varphi(\xi, \eta_1)\left[f(\eta_1) + \int_{\eta_1}^b H(\eta, \eta_1) f(\eta)\, d\sigma(\eta)\right] d\sigma(\eta_1) \quad (2.19)$$

Since

$$\left[f(\eta_1) + \int_{\eta_1}^b H(\eta, \eta_1) f(\eta)\, d\sigma(\eta)\right] \in L_{2,\sigma}$$

then, using Parseval's equality for φ-transform, realized by Equation (2.2)

$$\int_{-\infty}^{\infty} F^2(\xi)\, d\tau(\xi) = \int_a^b f^2(x)\, d\sigma(x)$$

we find

$$\int_{-\infty}^{\infty} \rho^2(\xi)\Phi^2(\xi)\, d\tau(\xi) = \int_a^b \left[f(\eta_1) + \int_{\eta_1}^b H(\eta, \eta_1) f(\eta)\, d\sigma(\eta)\right]^2 d\sigma(\eta_1) \quad (2.20)$$

Thus, on account of (2.18)

$$f(\eta_1) + \int_{\eta_1}^b H(\eta, \eta_1) f(\eta)\, d\sigma(\eta) = 0 \quad (2.21)$$

Equation (2.21) represents a regular ($f(\eta)$ is finite) Volterra equation relative to $f(\eta_1)$, which may have only the trivial solution $f(\eta_1) \equiv 0$. It follows that $\Phi(\xi) \equiv 0$.

To prove that the only solution is Equation (2.14) for every particular x, it is sufficient that the homogeneous equation

$$\psi(\eta_1) + \int_a^x \Psi(\eta, \eta_1)\,\psi(\eta)\,d\sigma(\eta) = 0 \tag{2.22}$$

has only a trivial solution. We suppose that there exists a function $\psi(\eta_1) \neq 0$ satisfying Equation (2.22). Substituting in (2.22), instead of $\psi(\eta, \eta_1)$ its value from (2.13) and changing the order of integration, we get

$$\int_{-\infty}^{\infty} \rho^2(\xi)\,u(\xi, \eta_1)\,d\tau(\xi) \int_a^x \psi(\eta)\,u(\xi, \eta)\,d\sigma(\eta) = 0 \tag{2.23}$$

Let $\psi_1(\eta)$ be a finite function

$$\psi_1(\eta) = \begin{cases} \psi(\eta) & (a < \eta \leqslant x) \\ 0 & (\eta > x) \end{cases}$$

Then

$$\int_{-\infty}^{\infty} \rho^2(\xi)\,u(\xi, \eta_1)\,d\tau(\xi) \int_a^b \psi_1(\eta)\,u(\xi, \eta)\,d\sigma(\eta) = 0 \tag{2.24}$$

or

$$\int_{-\infty}^{\infty} \rho^2(\xi)\,\Phi_1(\xi)\,u(\xi, \eta_1)\,d\tau(\xi) = 0 \tag{2.25}$$

where Φ_1 is a u-transform of function $\psi_1(\eta)$. Multiplying the left and right-hand sides of (2.25) by $\psi_1(\eta_1)$ and integrating with measure $\sigma(\eta_1)$ from a to b, we obtain

$$\int_{-\infty}^{\infty} \rho^2(\xi)\,\Phi^2(\xi)\,d\tau(\xi) = 0 \tag{2.26}$$

Hence, by virtue of (2.18) and of its consequence, it must be $\psi(\eta) \equiv 0$, which contradicts our assumption. Thus Equation (2.22) has only the trivial solution, consequently, integral equation (2.14) as a unique solution $K(x, \eta)$. Having determined $K(x, \eta)$ from Equation (2.14) by some well-known method, using Equation (2.3), we find the function $\varphi(\xi, x)$ and then from (2.5) we shall find the second orthogonalization kernel $H_0(x, \eta)$.

Thus, integral operators related to kernels $K(x, \eta)$ and $H(x, \eta)$ allow one to realize the orthogonalization of functions $\rho(\xi)u(\xi, \eta)$ and $\rho^{-1}(\xi)u(\xi, \eta)$ whereby, what is very important, integration is carried out in limits of each of the intervals, on which dual equations (2.1) are specified. This orthogonalization gives the possibility to reduce dual equations to a single integral equation, the solution of which at once follows from the inversion formula. Multiplying both sides of the first equation (2.1) by

$$K_0(x, \eta) = [\sigma'(\eta)]^{-1}\,\delta(x - \eta) + \begin{cases} K(x, \eta) & (\eta < x) \\ 0 & (\eta > x) \end{cases}$$

and both sides of the second by $H_0(x, \eta)$ and integrating with measure $\sigma(\eta)$

from a to b, we obtain

$$\int_{-\infty}^{\infty} f(\xi)\varphi(\xi, x)\,d\tau(\xi) = \begin{cases} G_1(x) & (a < x < c) \\ G_2(x) & (c < x < b) \end{cases} \tag{2.27}$$

$$G_1(x) = g_1(x) + \int_a^x K(x, \eta)\, g_1(\eta)\, d\sigma(\eta),$$

$$G_2(x) = g_2(x) + \int_x^b H(\eta, x)\, g_2(\eta)\, d\sigma(\eta)$$

Inverting (2.27), we find the solution of dual integral equations (2.1) in the form

$$f(\xi) = \int_a^c G_1(x)\varphi(\xi, x)\,d\sigma(x) + \int_c^b G_2(x)\varphi(\xi, x)\,d\sigma(x) \tag{2.28}$$

3. In the case of dual series

$$\begin{aligned} \sum_{n=0}^{\infty} \rho_n f_n u_n(\eta)\tau_n &= g_1(\eta) \qquad (a < \eta < c) \\ \sum_{n=0}^{\infty} \rho_n^{-1} f_n u_n(\eta)\tau_n &= g_2(\eta) \qquad (c < \eta < b) \end{aligned} \tag{3.1}$$

the method of solution is not different from that given in the previous Section. As was already noted, series (3.1) may be considered, as a particular case of dual equations (2.1), when $\tau(\xi)$ is represented by step functions. Thus we at once write the solution to dual series (3.1) in the form

$$\begin{aligned} f_n = &\int_a^c \left[u_n(x) + \int_a^x K(x, \eta_1)\, u_n(\eta_1)\, d\sigma(\eta_1)\right]\left[g_1(x) + \int_a^x K(x, \eta)\, g_1(\eta)\, d\sigma(\eta)\right] d\sigma(x) + \\ &+ \int_c^b \left[u_n(x) + \int_a^x K(x, \eta_1)\, u_n(\eta_1)\, d\sigma(\eta_1)\right]\left[g_2(x) + \int_x^b H(\eta, x)\, g_2(\eta)\, d\sigma(\eta)\right] d\sigma(x) \end{aligned} \tag{3.2}$$

Here

$$H(\eta, x) = \sum_{n=0}^{\infty} \rho_n u_n(\eta)\varphi_n(x)\tau_n - [\sigma'(\eta)]^{-1}\delta(x - \eta)$$

and $K(x, \eta)$ is a solution to the integral equation (2.14) for

$$\Psi(\eta, x) = \sum_{n=0}^{\infty} [\rho_n^2 - 1]\, u_n(x)\, u_n(\eta)\,\tau_n \tag{3.3}$$

For this, obviously, it is supposed that $u_n(x)$ is a system of functions, orthogonal with weight $\sigma'(x)$ and a normalized value of τ_n on the interval (a, b).

4. The proposed method may be used to solve also systems of dual integral equations. Let, for example, the given system be

$$\begin{aligned} \int_{-\infty}^{\infty} \rho(\xi)\,[f_1(\xi) + \alpha_1 f_2(\xi)]\, u(\xi, \eta)\, d\tau(\xi) &= g_1(\eta) \\ \int_{-\infty}^{\infty} \rho(\xi)\,[f_1(\xi) + \beta_1 f_2(\xi)]\, u(\xi, \eta)\, d\tau(\xi) &= g_3(\eta) \end{aligned} \qquad (a < \eta < c) \tag{4.1.1}$$

$$\int_{-\infty}^{\infty} \rho^{-1}(\xi)\,[f_1(\xi)+\alpha_2 f_2(\xi)]\,u(\xi,\eta)\,d\tau(\xi)=g_2(\eta)$$
$$\int_{-\infty}^{\infty} \rho^{-1}(\xi)\,[f_1(\xi)+\beta_2 f_2(\xi)]\,u(\xi,\eta)\,d\tau(\xi)=g_4(\eta) \qquad (c<\eta<b) \qquad (4.1.2)$$

Multiplying the left and right-hand sides of (4.1.1) by $K_0(x,\eta)$ and (4.1.2) by $H_0(x,\eta)$ and integrating with respect to measure $\sigma(\eta)$, we get

$$\int_{-\infty}^{\infty} [f_1(\xi)+\alpha_1 f_2(\xi)]\,\varphi(\xi,x)\,d\tau(\xi)=G_1(x)$$
$$\int_{-\infty}^{\infty} [f_1(\xi)+\beta_1 f_2(\xi)]\,\varphi(\xi,x)\,d\tau(\xi)=G_3(x) \qquad (a<x<c) \qquad (4.2.1)$$
$$\int_{-\infty}^{\infty} [f_1(\xi)+\alpha_2 f_2(\xi)]\,\varphi(\xi,x)\,d\tau(\xi)=G_2(x)$$
$$\int_{-\infty}^{\infty} [f_1(\xi)+\beta_2 f_2(\xi)]\,\varphi(\xi,x)\,d\tau(\xi)=G_4(x) \qquad (c<x<b) \qquad (4.2.2)$$

We denote

$$F_n(x)=\int_{-\infty}^{\infty} f_n(\xi)\,\varphi(\xi,x)\,d\tau(\xi) \qquad (n=1,2)$$

then from (4.2.1) we have

$$F_1(x)=\frac{\alpha_1 G_3(x)-\beta_1 G_1(x)}{\alpha_1-\beta_1}, \qquad F_2(x)=\frac{G_1(x)-G_3(x)}{\alpha_1-\beta_1} \qquad (a<x<c) \qquad (4.3.1)$$

and from (4.2.2)

$$F_1(x)=\frac{\alpha_2 G_4(x)-\beta_2 G_2(x)}{\alpha_2-\beta_2}, \qquad F_2(x)=\frac{G_2(x)-G_4(x)}{\alpha_2-\beta_2} \qquad (c<x<b) \qquad (4.3.2)$$

Transforming (4.3), we find the solution to the system (4.1)

$$f_1(\xi)=\frac{1}{\alpha_1-\beta_1}\int_a^c [\alpha_1 G_3(x)-\beta_1 G_1(x)]\,\varphi(\xi,x)\,d\sigma(x)+ \qquad (4.4)$$
$$+\frac{1}{\alpha_2-\beta_2}\int_c^b [\alpha_2 G_4(x)-\beta_2 G_2(x)]\,\varphi(\xi,x)\,d\sigma(x)$$

$$f_2(\xi)=\frac{1}{\alpha_1-\beta_1}\int_a^c [G_1(x)-G_3(x)]\,\varphi(\xi,x)\,d\sigma(x)+ \qquad (4.5)$$
$$+\frac{1}{\alpha_2-\beta_2}\int_c^b [G_2(x)-G_4(x)]\,\varphi(\xi,x)\,d\sigma(x)$$

5. The results obtained above relate to dual integral equations with orthogonal kernels, for which Equations (2.2) are valid, ensuring the existence of integral equation (2.14) for the determination of the orthogonalization kernel. However, carrying out the orthogonalization and obtaining integral equation of the type (2.14) is possible also in the more general case, namely for all those equations, which on the interval (a, b) allow inversion. For Equations (1.1), and also, for example, for dual equations of a more general type

$$\int_{(L)} [1 + h(\xi)] \psi(\xi) u(\xi, \eta) d\tau(\xi) = g_1(\eta) \qquad (a < \eta < c)$$
$$\int_{(L)} \psi(\xi) u(\xi, \eta) d\tau(\xi) = g_2(\eta) \qquad (c < \eta < b) \tag{5.1}$$

where (L) is a certain contour in the complex plane, the solution by the proposed method may be obtained in those cases, when for Equation

$$\int_{(L)} f(\xi) u(\xi, \eta) d\tau(\xi) = g(\eta) \qquad (a < \eta < b) \tag{5.2}$$

there is known the inversion formula

$$f(\xi) = \int_a^b g(\eta) v(\xi, \eta) d\sigma(\eta) \tag{5.3}$$

Relations (5.2) and (5.3) hold for integral transforms with nonsymmetrical kernels and in particular for expansions into eigenfunctions of nonself-adjoint differential operators and also in certain special cases. For example, it is known [3] that for certain conditions integral equations with a kernel, depending upon product arguments

$$\int_0^\infty F(\xi) k_1(\xi x) d\xi = f(x) \qquad (0 \leqslant x < \infty) \tag{5.4}$$

have a solution

$$F(\xi) = \int_0^\infty f(x) k_2(\xi x) dx \tag{5.5}$$

$$k_2(\xi x) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{(\xi x)^{-s}}{\mathrm{K}_1(1-s)} ds \qquad \left(\mathrm{K}_1(1-s) = \int_0^\infty k_1(x) x^{-s} dx\right) \tag{5.6}$$

Here $\mathrm{K}_1(1-s)$ denotes a Mellin transform of the function $k_1(x)$

Analogous relations are easily established also for the equations given along the whole axis, with kernels depending on the difference or sum of the arguments

$$\int_{-\infty}^{\infty} F(\xi) k_1(x \pm \xi) d\xi = f(x) \tag{5.7}$$

The inverse equation has the form

$$F(\xi) = \int_{-\infty}^{\infty} f(x) k_2(x \pm \xi) dx \qquad \left(k_2(x \pm \xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\exp[ip(x \pm \xi)]}{k_1^*(p)} dp\right) \tag{5.8}$$

Here $k_1^*(p)$ is the Fourier transform of the function $k_1(x)$.

Substituting (5.3) into (5.2) and conversely, we find

$$\sigma'(\eta)\int_{(L)} u(\xi,\eta)\,v(\xi,\eta_1)\,d\tau(\xi)=\delta(\eta-\eta_1),$$

$$\tau'(\xi)\int_a^b u(\xi,\eta)\,v(\xi_1,\eta)\,d\sigma(\eta)=\delta(\xi-\xi_1) \qquad (5.9)$$

Orthogonalizing kernels $K(x,\eta)$ and $H(x,\eta)$ are found from the relations

$$\rho^{-1}(\xi)\varphi(\xi,x)=u(\xi,x)+\int_a^x K(x,\eta)\,u(\xi,\eta)\,d\sigma(\eta) \qquad (5.10)$$

$$\rho(\xi)\varphi(\xi,x)=u(\xi,x)+\int_x^b H(\eta,x)\,u(\xi,\eta)\,d\sigma(\eta) \qquad (5.11)$$

Here, as above, $\rho(\xi)=\sqrt{1+h(\xi)}$. Multiplying (5.11) by $v(\xi,\eta_1)$ and integrating with respect to $\tau(\xi)$ along the contour (L), we find

$$\int_{(L)} \rho(\xi)\varphi(\xi,x)\,v(\xi,\eta_1)\,d\tau(\xi)=\frac{\delta(x-\eta_1)}{\sigma'(\eta_1)}+\begin{cases}H(\eta_1,x) & (x<\eta_1)\\ 0 & (x>\eta_1)\end{cases} \qquad (5.12)$$

Using now (5.10), we find for $x>\eta_1$

$$\Psi_1(x,\eta_1)+\int_a^x K(x,\eta)\,\Psi_1(\eta,\eta_1)\,d\sigma(\eta)+K(x,\eta_1)=0 \qquad (5.13)$$

where

$$\Psi_1(x,\eta_1)=\int_{(L)} h(\xi)\,u(\xi,x)\,v(\xi,\eta_1)\,d\tau(\xi) \qquad (5.14)$$

6. In individual cases one may be able to realize orthogonalization and to determine the orthogonalization kernels without the solution of integral equations (2.14) or (5.13). If, proceeding from some consideration, one may predict the form of function $\varphi(\xi,x)$, then kernel $K_0(x,\eta)$ is determined by Equation

$$K_0(x,\eta)=\int_{(L)} \rho^{-1}(\xi)\varphi(\xi,x)\,v(\xi,\eta)\,d\tau(\xi) \qquad (6.1)$$

obtained by means of the inversion of (5.10) and the kernel $H_0(x,\eta)$ by Equation

$$H_0(x,\eta)=\int_{(L)} \rho(\xi)\varphi(\xi,x)\,v(\xi,\eta)\,d\tau(\xi) \qquad (6.2)$$

Analogous relations occur for kernels $K_0(x,\eta)$ and $H_0(x,\eta)$ in the case of dual equations (2.1).

7. As an illustration, we consider the plane contact problem in the theory of elasticity for an infinite wedge in polar coordinates. Let the end of the wedge, the angle of which equals 2α, on section $0<r<1$ be pressed without friction by symmetrical rigid stamps, so that boundary conditions for the problem have the following form:

$$\begin{aligned} &v(r,\pm\alpha)=g(r) && (0<r<1)\\ &\sigma_\theta(r,\pm\alpha)=0 && (1<r<\infty)\\ &\tau_{r\theta}(r,\pm\alpha)=0 && (0<r<\infty)\end{aligned} \qquad (7.1)$$

As is known, the problems of a wedge are successfully solved by the help of Mellin transforms. Introducing Mellin transforms $\sigma_r^*, \sigma_\theta^*, \tau_{r\theta}^*$ for stresses $\sigma_r, \sigma_\theta, \tau_{r\theta}$, and supposing that the stresses are of order $r^{-\varepsilon_1}$ $(\varepsilon_1>1)$ for $r\to\infty$ and $r^{-\varepsilon_2}$ $(\varepsilon_2<1)$ for $r\to 0$, from the equations of equilibrium and the condition of continuity one may obtain ordinary differential equations of the fourth order with respect to σ_θ^*, the solution of which has the form [24]

$$\sigma_\theta^* = A\cos(p+1)\theta + B\cos(p-1)\theta + C\sin(p+1)\theta + D\sin(p-1)\theta \quad (7.2)$$

where

$$\sigma_r^* = \frac{1}{p}\left(\frac{\sigma_\theta^{*''}}{p-1} - \sigma_\theta^*\right), \qquad \tau_{r\theta}^* = \frac{1}{p-1}\sigma_\theta^{*'} \quad (7.3)$$

Here $p\ (\varepsilon_2 - 1 < \operatorname{Re} p < \varepsilon_1 - 1)$ is a complex parameter in the Mellin transform. From symmetry conditions of the stresses relative to $\theta = 0$ it follows that $C = D = 0$. Using (7.3) and the last condition in (7.1), we find in addition

$$B(p) = -A(p)\frac{(p+1)\sin(p+1)\alpha}{(p-1)\sin(p-1)\alpha} \quad (7.4)$$

Since (for generalized plane state of stress)

$$u = \frac{1}{E}\int_r^\infty (\sigma_r - \nu\sigma_\theta)\,dr, \qquad v = \frac{r}{E}\int_0^\theta (\sigma_\theta - \nu\sigma_r)\,d\theta - \int_0^\theta u\,d\theta \quad (7.5)$$

where ν is Poisson's coefficient, then using (7.2) and (7.3) we obtain, after conversion of transforms σ_θ^* and σ_r^* in accordance with (7.1), the following dual integral equation with respect to the function $A_1(p)$

$$\frac{1}{2\pi i}\int_{(L)} A_1(p)\rho(p)\frac{dp}{r^p} = -Eg(r) \quad (0 < r < 1), \quad \frac{1}{2\pi i}\int_{(L)} \frac{A_1(p)\,dp}{r^p\rho(p)} = 0 \quad (1 < r < \infty)$$

$$A_1(p) = A(p)f_0(p)f_1(p), \qquad f_0(p) = \frac{2\left[(p^2+p+\nu-1)\sin(p+1)\alpha\right]^{1/2}}{p^2-1}$$

$$\rho(p) = \frac{f_0(p)}{f_1(p)}, \quad f_1(p) = \left[\cos(p+1)\alpha - \frac{(p+1)\sin(p+1)\alpha}{(p-1)\sin(p-1)\alpha}\cos(p-1)\alpha\right]^{1/2} \quad (7.6)$$

Here (L) is a vertical straight line inside the strip $(\varepsilon_2 - 1 < \operatorname{Re} p < \varepsilon_1 - 1)$.

For solution of dual equations (7.6) one should find the orthogonalization kernel $K(x, r)$ from Equation

$$\Psi(x, r) + \int_0^x K(x, r_1)\Psi(r, r_1)\,dr_1 + K(x, r) = 0 \quad (7.7)$$

where

$$\Psi(x, r) = \frac{1}{2\pi i}\int_{(L)} [\rho^2(p) - 1]\,x^{-p}r^{p-1}\,dp \quad (7.8)$$

After the determination of kernel $K(x, r)$ from (7.7) the solution of dual Equations (7.6) is written in the form

$$A_1(p) = -E\rho(p)\int_0^1 \left[x^{p-1} + \int_0^x K(x, x_1)x_1^{p-1}\,dx_1\right]\left[g_1(x) + \int_0^x K(x, r)g_1(r)\,dr\right]dx \quad (7.9)$$

and the stresses may be determined with the aid of the Mellin inversion formulas from (7.2) and (7.3).

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